1.4 Elliptic operator

1.4.1 Parametrix

Definition 1.4.1. Let P be a differential operator. We say P is elliptic if for any non-zero cotangent vector $\xi \in T^*M$, the principal symbol $\sigma_\xi(P) : E \to F$ is invertible.

Definition 1.4.2. Let $P \in \Psi\mathrm{DO}_m$ with symbol p. We say P is elliptic if there exists a constant $c > 0$ such that for all $|\xi| \geq c$, the matrix inverse of $p(x,\xi)$ exists and satisfies

$$|p(x,\xi)^{-1}| \leq C(1+|\xi|)^{-m} \tag{1.4.1}$$

for some constant $C > 0$.

Definition 1.4.3. Let $P \in \Psi\mathrm{DO}_m(E,F)$. We say P is elliptic if its principal symbol $\sigma_\xi(P) \in \mathrm{Sym}^m(E,F)/\mathrm{Sym}^{m-1}(E,F)$ has a representative p and for some Riemannian metric, there exists a constant $c > 0$, such that for all $|\xi| \geq c$, the matrix inverse of $p(x,\xi)$ exists and satisfies

$$|p(\xi)^{-1}| \leq C(1+|\xi|)^{-m} \tag{1.4.2}$$

for some constant $C > 0$.

Remark that the symbol $p(x,\xi)$ in conditions (1.4.1) and (1.4.2) could be replaced by the principal symbol $\sigma_\xi(P)$.

It is easy to see that if P is a differential operator, Definition 1.4.3 is equivalent to Definition 1.4.1.

From Theorem 1.3.13, if $P \in \Psi\mathrm{DO}_m(E,F)$, $P^* \in \Psi\mathrm{DO}_m(F^*,E^*)$. Since $\sigma_\xi(P^*) = \overline{\sigma_\xi(P)}^T$, P is elliptic if and only if P^* is elliptic.

Lemma 1.4.4. *Let $P \in \Psi\mathrm{DO}_m$ be a elliptic operator. Then there exists an operator $Q \in \Psi\mathrm{DO}_{-m}$, unique up to equivalence, such that*

$$PQ = \mathrm{Id} - S', \quad QP = \mathrm{Id} - S, \tag{1.4.3}$$

where $S, S' \in \Psi\mathrm{DO}_{-\infty}$.

Proof. We only need to prove that for any compact subset $K \subset \mathbb{R}^n$, on $\mathscr{C}_0^\infty(M,E)$, (1.4.3) holds. Since the composition of a smoothing operator and a pseudodifferential operator is a smoothing operator, by Proposition 1.3.8, we may assume that P and Q have compact support.

Let c be the constant in Definition 1.4.2. Let $\chi : \mathbb{R}^+ \to [0,1]$ be a smooth cut-off function such that $\chi(t) = 0$ for $t \le c$ and $\chi(t) = 1$ for $t \ge 2c$. For $x \in K$, set

$$q_0(x,\xi) = \chi(|\xi|)p(x,\xi)^{-1}. \tag{1.4.4}$$

We claim that

$$q_0 \in \operatorname{Sym}^{-m}. \tag{1.4.5}$$

In fact, for $\alpha = \beta = 0$, from (1.4.1), on $|\xi| \ge c$, we have

$$|D_x^\alpha D_\xi^\beta p(x,\xi)^{-1}| \le C(1+|\xi|)^{-m-|\beta|}. \tag{1.4.6}$$

We write $\gamma = (\alpha,\beta) \in \mathbb{N}^{2n}$ and $D^\gamma = D_x^\alpha D_\xi^\beta$ On $|\xi| \ge c$, we have

$$0 = D^\gamma(p \cdot p^{-1}) = \sum_{\gamma'+\gamma''=\gamma} \frac{\gamma!}{\gamma'!\gamma''!} D^{\gamma'}p \cdot D^{\gamma''}(p^{-1}). \tag{1.4.7}$$

Thus we have

$$D^\gamma p^{-1} = -p^{-1} \cdot \sum_{\gamma'+\gamma''=\gamma, \gamma'\ne 0} \frac{\gamma!}{\gamma'!\gamma''!} D^{\gamma'}p \cdot D^{\gamma''}(p^{-1}). \tag{1.4.8}$$

Now we take induction on $|\gamma|$. If $|\gamma| = 0$, (1.4.6) holds. We assume that (1.4.6) holds for $|\gamma| \le k-1$. Then by (1.4.8), we have

$$\begin{aligned} |D^\gamma(p^{-1})| \le C \sum_{|\beta'|+|\beta''|=|\beta|} &(1+|\xi|)^{-m}(1+|\xi|)^{m-|\beta'|}(1+|\xi|)^{-m-|\beta''|} \\ &\le C(1+|\xi|)^{-m-|\beta|}. \end{aligned} \tag{1.4.9}$$

Therefore, on $|\xi| \ge c$, (1.4.6) holds for any α, β. Thus the claim (1.4.5) follows from

$$|D^\gamma q(x,\xi)| \le \sum_{\gamma'+\gamma''=\gamma} \frac{\gamma!}{\gamma'!\gamma''!} |D^{\gamma'}\chi||D^{\gamma''}(p^{-1})| \le C(1+|\xi|)^{m-\beta}. \tag{1.4.10}$$

By induction, we set

$$q_k = -\sum_{j=0}^{k-1} \left\{ \sum_{|\alpha|+j=k} \frac{i^{|\alpha|}}{\alpha!} (D_\xi^\alpha q_j)(D_x^\alpha p) \right\} \cdot q_0. \tag{1.4.11}$$

From (1.4.5), we have

$$q_k \in \mathrm{Sym}^{m-k}\,. \tag{1.4.12}$$

Let $Q_k \in \Psi\mathrm{DO}_{m-k}$ be the pseudodifferential operator with symbol p_k. Then we have

$$\begin{aligned}
\mathrm{Sym}(Q_0P) &\sim 1 + \sum_{|\alpha|=1} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_0)(D_x^\alpha p) + \sum_{|\alpha|=2} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_0)(D_x^\alpha p) + \cdots\\
\mathrm{Sym}(Q_1P) &\sim -\sum_{|\alpha|=1} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_0)(D_x^\alpha p) q_0 p + \sum_{|\alpha|=1} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_1)(D_x^\alpha p) + \cdots\\
\mathrm{Sym}(Q_2P) &\sim -\sum_{j=0}^{1}\left\{ \sum_{|\alpha|+j=2} \frac{i^{|\alpha|}}{\alpha!}(D_\xi^\alpha q_j)(D_x^\alpha p)\right\}\cdot q_0 p_0 + \cdots\\
&\cdots\ \cdots
\end{aligned} \tag{1.4.13}$$

From (1.4.13), letting

$$q \sim \sum q_k, \tag{1.4.14}$$

and $Q \in \Psi\mathrm{DO}_{-m}$ be the pseudodifferential operator with symbol q, we have $\mathrm{Sym}(QP) = 1$. Thus $QP - \mathrm{Id}$ is a smoothing operator. Remark that all constants in this proof are independent of the compact subset.

From the same argument, we could construct $Q' \in \Psi\mathrm{DO}_{-m}$ such that $PQ' - \mathrm{Id}$ is a smoothing operator. Thus

$$Q \sim Q(PQ') \sim (QP)Q' \sim Q'. \tag{1.4.15}$$

The proof of Lemma 1.4.4 is completed. □

Theorem 1.4.5. *Assume that $P \in \Psi\mathrm{DO}_m(E,F)$ is elliptic. Then there exists $Q \in \Psi\mathrm{DO}_{-m}(F,E)$, up to equivalence, such that*

$$PQ = \mathrm{Id} - S', \quad QP = \mathrm{Id} - S, \tag{1.4.16}$$

*where S, S' are smoothing operators. The operator Q is called a **parametrix** of P.*

Proof. We take a coordinate system $\{U_i, \phi_i\}$ such that E, F are trivial on U_i. Let $\{\psi_i\}$ be a partition of unity with respect to $\{U_i\}$. By Definition 1.4.3, for any compact subset K of U_i, $P : \mathscr{C}_0^\infty(K,E) \to \mathscr{C}^\infty(K,F)$ is elliptic. By

Lemma 1.4.4, there exists $Q_i \in \Psi\mathrm{DO}_{-m} : \mathscr{C}_0^\infty(U_i, E) \to \mathscr{C}^\infty(U_i, F)$, such that $PQ_i = \mathrm{Id} - S_i$, where $S_i : \mathscr{C}_0^\infty(U_i, E) \to \mathscr{C}^\infty(U_i, F)$ is a smoothing operator. As in the proof of Lemma 1.4.4, we may assume that $Q_i\psi_i$ has compact support K_i such that $\mathrm{supp}(\psi_i) \subset K_i \subset U_i$. Let $\varphi, \varphi_i \in \mathscr{C}_0^\infty(U_i)$ such that $\varphi \equiv 1$ on K_i and $\varphi_i \equiv 1$ on $\mathrm{supp}(\varphi)$. Then

$$\varphi_i P \varphi Q_i \psi_i = \varphi_i P Q_i \psi_i = \psi_i - \varphi_i S_i \psi_i. \tag{1.4.17}$$

Note that $\varphi_i S_i \psi_i \in \Psi\mathrm{DO}_{-\infty}(E, F)$. By Proposition 1.3.18 (2), $(1-\varphi_i)PQ_i\psi_i = (1-\varphi_i)P\varphi Q_i\psi_i \in \Psi\mathrm{DO}_{-\infty}(E, F)$. Thus

$$\begin{aligned} Q' &:= \sum Q_i \psi_i \in \Psi\mathrm{DO}_{-m}(E, F), \\ S' &:= \sum \varphi_i S_i \psi_i + \sum (1-\varphi_i) P Q_i \psi_i \in \Psi\mathrm{DO}_{-\infty}(E, F). \end{aligned} \tag{1.4.18}$$

Then

$$\begin{aligned} PQ' = P(\sum Q_i\psi_i) &= \sum PQ_i\psi_i = \sum \varphi_i PQ_i\psi_i + \sum (1-\varphi_i)PQ_i\psi_i \\ &= \sum \psi_i - \sum \varphi_i S_i \psi_i + \sum (1-\varphi_i) P Q_i \psi_i = \mathrm{Id} - S'. \end{aligned} \tag{1.4.19}$$

From (1.4.19), For $P^* \in \Psi\mathrm{DO}_m(F^*, E^*)$, there exists $Q^* \in \Psi\mathrm{DO}_{-m}(E^*, F^*)$ such that $P^*Q^* = \mathrm{Id} - S^*$, where $S \in \Psi\mathrm{DO}_{-\infty}(E^*, E^*)$. Taking the adjoint, we have $QP = \mathrm{Id} - S$. As in (1.4.15), we have $Q \sim Q'$.

The proof of Theorem 1.4.5 is completed. □

Let $L^2_{loc}(M, E)$ be the space of locally L^2-integrable sections of E on X (L^2-integrable on any bounded subset of X).

Theorem 1.4.6 (Elliptic regularity)**.** *Let $P \in \Psi\mathrm{DO}_m(E, F)$ be a elliptic operator.*

(1)Let $u \in L^2_{loc}(M, E)$ with compact support K. If $Pu \in \mathbf{H}_0^s(K, F)$, then $u \in \mathbf{H}_0^{s+m}(K, E)$.

(2) For any open subset $U \subset M$, if $Pu \in \mathscr{C}^\infty(U, F)$, then $u \in \mathscr{C}^\infty(U, E)$.

(3) If $Pu = \lambda u$ for some $\lambda \in \mathbb{C}$ and $m > 0$, then u is smooth.

Proof. (1) From Theorem 1.4.5, there exists $Q \in \Psi\mathrm{DO}_{-m}(F, E)$, such that $\mathrm{Id} = QP + S$, where S is a smoothing operator. So

$$\|u\|_{s+m} \le \|QPu\|_{s+m} + \|Su\|_{s+m} \le C\|Pu\|_s + C\|u\|_0 \le \infty. \tag{1.4.20}$$

Thus $u \in \mathbf{H}_0^{s+m}(K, E)$.

(2) For $U \subset M$, if $Pu \in \mathscr{C}^\infty(U, F)$, by Proposition 1.3.17, $QPu \in \mathscr{C}^\infty(U, E)$ and $Su \in \mathscr{C}^\infty(U, E)$. Thus by $\mathrm{Id} = QP + S$, $u \in \mathscr{C}^\infty(U, E)$.

(3) If $m > 0$, $P - \lambda\,\mathrm{Id} \in \Psi\mathrm{DO}_m(E, F)$. By (2), since $(P - \lambda\,\mathrm{Id})u = 0$ is smooth, u is smooth.

The proof of Theorem 1.4.6 is completed. □

Theorem 1.4.7 (Fundamental elliptic estimate)**.** *Let $P \in \Psi\mathrm{DO}_m(E, F)$ be a elliptic operator. For any $s \in \mathbb{R}$, there exists $C > 0$ such that for any $u \in \mathbf{H}_0^{s+m}(K, E)$, we have*

$$\|u\|_{s+m} \leq C(\|Pu\|_s + \|u\|_s). \tag{1.4.21}$$

Thus the norms $\|\cdot\|_{s+m}$ and $\|P\cdot\|_s + \|\cdot\|_s$ are equivalent.

Proof. From Theorem 1.4.5, there exists $Q \in \Psi\mathrm{DO}_{-m}(F, E)$, such that $\mathrm{Id} = QP + S$, where S is a smoothing operator. So

$$\|u\|_{s+m} \leq \|QPu\|_{s+m} + \|Su\|_{s+m} \leq C\|Pu\|_s + C\|u\|_s. \tag{1.4.22}$$

The proof of Theorem 1.4.7 is completed. □

Obviously, Theorems 1.4.6 and 1.4.7 hold for $M = \mathbb{R}^n$. We state it here.

Theorem 1.4.8. *Let $P \in \Psi\mathrm{DO}_m$ be a elliptic operator.*

(1)Let $u \in L^2$. If $Pu \in \mathbf{H}^s$, then $u \in \mathbf{H}^{s+m}$.

(2) For any open subset $U \subset \mathbb{R}^n$, if $Pu \in \mathscr{C}^\infty(U, \mathbb{C}^p)$, then $u \in \mathscr{C}^\infty(U, \mathbb{C}^p)$.

(3) If $Pu = \lambda u$ for some $\lambda \in \mathbb{C}$ and $m > 0$, then u is smooth.

(4) For any $s \in \mathbb{R}$, there exists $C, C' > 0$ such that for any $u \in \mathbf{H}^{s+m}$, we have

$$\|u\|_{s+m} \leq C(\|Pu\|_s + \|u\|_s) \leq C'\|u\|_{s+m}. \tag{1.4.23}$$

Thus the norms $\|\cdot\|_{s+m}$ and $\|P\cdot\|_s + \|\cdot\|_s$ are equivalent.

Corollary 1.4.9 (Inner gradient estimate)**.** *Let P be an elliptic differential operator of order $m > 0$ defined on an open subset Ω of $\mathbb{R}^n$. Then for any compact subset $K \subset \Omega$ and $k \in \mathbb{N}$, there exists $C_{K,k} > 0$ such that for any solution u of the equation $Pu = 0$, we have*

$$\|u\|_{K,\mathscr{C}^k} \leq C_{K,k}\|u\|_{\Omega,\mathscr{C}^0}, \quad \|u\|_{K,\mathscr{C}^k} \leq C_{K,k}\|u\|_{\Omega,L^2}. \tag{1.4.24}$$

Proof. Choose $\varphi \in \mathscr{C}_0^\infty(\Omega)$ such that $\varphi \equiv 1$ on K. Then

$$P(\varphi u) = \varphi Pu + P'u = P'u, \tag{1.4.25}$$

where P' is a differential operator of order $m - 1$. By Theorem 1.4.7,

$$\begin{aligned}\|u\|_{K,s} \leq \|\varphi u\|_{\Omega,s} &\leq C(\|P(\varphi u)\|_{\Omega,s-m} + \|\varphi u\|_{\Omega,s-m}) \\ &= C(\|P'u\|_{\Omega,s-m} + \|\varphi u\|_{\Omega,s-m}) \leq C'\|u\|_{\Omega,s-1}.\end{aligned} \tag{1.4.26}$$

Take a sequence $K \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset \Omega_N = \Omega$. Using (1.4.26) repeatedly, by Sobolev embedding theorem, we get the inner gradient estimate (1.4.24).

The proof of Corollary 1.4.9 is completed. □

1.4.2 Laplacian

In this subsection, we introduce the most important elliptic operator: Laplacian.

For vector fields X, Y on manifold M, we define the vector field $[X, Y]$ by

$$[X,Y]f = X(Yf) - Y(Xf), \tag{1.4.27}$$

for any $f \in \mathscr{C}^\infty(M)$. Assume that M is a Riemannian manifold with Riemannian metric g^{TM}. Then there is a canonical connection $\nabla^{TM} : \mathscr{C}^\infty(M, TM) \to \mathscr{C}^\infty(T^*M \otimes TM)$, called the **Levi-Civita connection** defined by

$$\begin{aligned} 2(\nabla_X Y, Z) = ([X,Y],Z) - ([Y,Z],X) + ([Z,X],Y) \\ + X(Y,Z) + Y(Z,X) - Z(X,Y), \end{aligned} \tag{1.4.28}$$

where X, Y, Z are vector fields and $(\cdot,\cdot) = g^{TM}(\cdot,\cdot)$. It is the unique connection which preserves the Riemannian metric:

$$d(X,Y) = (\nabla X, Y) + (X, \nabla Y) \tag{1.4.29}$$

for vector fields X, Y, and which is torsion free:

$$\nabla_X Y - \nabla_Y X = [X,Y]. \tag{1.4.30}$$

On $\mathbb{R}^n$, the Laplace operator is defined by

$$\Delta = -\frac{\partial^2}{\partial x_i^2}. \tag{1.4.31}$$

Let E be a vector bundle over a Riemannian manifold M, with connection ∇^E. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of (T_xM, g^{TM}), i.e., locally e_i is a vector field and at $x \in M$, $(e_i, e_j) = \delta_{ij}$. Then naively we want to define the Laplace operator on vector bundles by

$$\Delta^E = -\nabla^E_{e_i}\nabla^E_{e_i}. \tag{1.4.32}$$

As usual, we need to check that the operator is independent of the basis chosen. Let $\{e'_i\}_{i=1}^n$ be an orthonormal basis of (T_xM, g^{TM}) such that $e'_i = h_{ij}e_j$, where (h_{ij}) is an orthogonal matrix. Then using this basis,

$$\begin{aligned} -\nabla^E_{e'_i}\nabla^E_{e'_i} &= -h_{ik}\nabla^E_{e_k}(h_{ij}\nabla^E_{e_j}) = -h_{ik}h_{ij}\nabla^E_{e_k}\nabla_{e_j} - h_{ik}\nabla^E_{e_k(h_{ij})e_j} \\ &= -\delta_{jk}\nabla^E_{e_k}\nabla_{e_j} - h_{ik}\nabla^E_{\nabla_{e_k}(h_{ij}e_j) - h_{ij}\nabla_{e_k}e_j} \\ &= -\nabla^E_{e_i}\nabla^E_{e_i} - \nabla^E_{\nabla_{e'_i}e'_i} + \nabla^E_{\nabla_{e_i}e_i}, \end{aligned} \tag{1.4.33}$$

where ∇ is any connection on TM. Thus

$$-\nabla^E_{e_i}\nabla^E_{e_i} + \nabla^E_{\nabla_{e_i}e_i} \tag{1.4.34}$$

does not depend on the choice of the basis. For the convenience and the uniqueness, we take $\nabla = \nabla^{TM}$, the Levi-Civita connection.

Definition 1.4.10. The Laplacian Δ^E on $\mathscr{C}^\infty(M,E)$ is the second order differential operator

$$\Delta^E = -\nabla^E_{e_i}\nabla^E_{e_i} + \nabla^E_{\nabla^{TM}_{e_i}e_i}. \tag{1.4.35}$$

Locally, the second order term of the Laplacian is just (1.4.31). So the principal symbol

$$\sigma_\xi(\Delta^E) = |\xi|^2 \cdot \mathrm{Id}_E\,. \tag{1.4.36}$$

By Definition 1.4.1, Δ^E is a elliptic operator.

Proposition 1.4.11. *For $u_1, u_2 \in \mathscr{C}^\infty_0(M,E)$, we have*

$$\int_M (\Delta^E u_1, u_2)dv = \int_M (\nabla^E_{e_i}u_1, \nabla^E_{e_i}u_2)dv = \int_M (u_1, \Delta^E u_2)dv. \tag{1.4.37}$$

Proof. Take $\alpha \in \mathscr{C}^\infty(M, T^*M)$ such that

$$\alpha(X) = (\nabla^E_X u_1, u_2), \tag{1.4.38}$$

for vector field X. Then

$$e_i(\alpha(e_i)) = (\nabla^E_{e_i}u_1, \nabla^E_{e_i}u_2) - (\Delta u_1, u_2) + \alpha(\nabla^{TM}_{e_i}e_i). \tag{1.4.39}$$

Note that $\alpha(e_i)e_i$ is a vector field which is independent of the choice of the basis. Then

$$\beta = i_{\alpha(e_i)e_i}dv = \sum_i (-1)^i \alpha(e_i)e^1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e^n. \tag{1.4.40}$$

is a global defined differential form. Since $e^i \wedge \nabla^{T^*M}_{e_i}$ satisfies three conditions in Proposition 1.1.11 by replacing d to $e^i \wedge \nabla^{T^*M}_{e_i}$, we have an important formula

$$d = e^i \wedge \nabla^{T^*M}_{e_i}. \tag{1.4.41}$$

Thus

$$d\beta = e_i(\alpha(e_i))e^1 \wedge \cdots \wedge e^n + \sum_{i,k}(-1)^i \alpha(e_i) e^k \wedge \nabla^{T^*M}_{e_k}(e^1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e^n)$$
$$= e_i(\alpha(e_i))e^1 \wedge \cdots \wedge e^n + \sum_i \alpha(e_i)\nabla^{T^*M}_{e_i}(e^1 \wedge \cdots \wedge e^n)$$
$$- \sum_{i,k} \alpha(e_i)(\nabla^{T^*M}_{e_k} e^k, e_i) e^1 \wedge \cdots \wedge e^n. \quad (1.4.42)$$

Note that

$$\langle \nabla_{e_i} e^k, e^k \rangle = \frac{1}{2}(\langle \nabla_{e_i} e^k, e^k \rangle + \langle e^k, \nabla_{e_i} e^k \rangle) = e_i(\langle e^k, e^k \rangle) = e_i(1) = 0. \quad (1.4.43)$$

Thus

$$\nabla^{T^*M}_{e_i}(e^1 \wedge \cdots \wedge e^n) = \sum_k \langle \nabla_{e_i} e^k, e^k \rangle e^1 \wedge \cdots \wedge e^n = 0. \quad (1.4.44)$$

Since

$$(\nabla^{T^*M}_{e_k} e^k, e_i) = -\langle e_k, \nabla^{TM}_{e_k} e_i \rangle = \langle \nabla^{TM}_{e_k} e^k, e_i \rangle, \quad (1.4.45)$$

from (1.4.42)-(1.4.44), we have

$$d\beta = (e_i(\alpha(e_i)) - \alpha(\nabla^{TM}_{e_i} e_i))e^1 \wedge \cdots \wedge e^n. \quad (1.4.46)$$

From (1.4.39) and (1.4.46), we have

$$(\nabla^E_{e_i} u_1, \nabla^E_{e_i} u_2) dv - (\Delta u_1, u_2) dv = d\beta. \quad (1.4.47)$$

Therefore, our proposition follows from (1.4.47) and the Stokes formula.

The proof of Proposition 1.4.11 is completed. □

Definition 1.4.12. The **generalized Laplacian** H associated with ∇^E is of the form

$$H = \Delta^E + Q, \quad (1.4.48)$$

where Q is a Hermitian section of $\mathrm{End}(E) = E^* \otimes E$ on M with lower bound, i.e., $Q^*(x) = Q(x)$ for $x \in M$, and there exists $C > 0$, such that for any $u \in \mathscr{C}_0^\infty(M, E)$,

$$(Qu, u)_{L^2} \geq -C\|u\|^2. \quad (1.4.49)$$

From Proposition 1.4.11, we have

$$\int_M (Hu_1, u_2)dv = \int_M (u_1, Hu_2)dv. \tag{1.4.50}$$

Theorem 1.4.13 (Gårding's inequality)**.** *Let $K \subset M$ be a compact subset. There exists $C > 0$ such that for any $u \in \mathscr{C}_0^\infty(K, E)$,*

$$\|u\|_1^2 \leq C\big((Hu, u)_{L^2} + \|u\|_0^2\big). \tag{1.4.51}$$

Proof. From Proposition 1.4.11 (1.4.48)and (1.4.49), we have

$$\begin{aligned}(Hu, u)_{L^2} = (\Delta u, u)_{L^2} + (Qu, u)_{L^2} = \|u\|_1^2 + (Qu, u)_{L^2} - \|u\|_0^2 \\ \geq \|u\|_1^2 - (C+1)\|u\|_0^2.\end{aligned} \tag{1.4.52}$$

The proof of Theorem 1.4.13 is completed. □

1.4.3 Fredholm operator

Let $T : H_1 \to H_2$ be a bounded linear map between Hilbert spaces. The kernel of T is

$$\mathrm{Ker}(T) := \{v \in H_1 : Tv = 0\}. \tag{1.4.53}$$

The range of T is

$$\mathrm{Im}(T) := \{Tv \in H_2 : v \in H_1\}. \tag{1.4.54}$$

The cokernel of T is the quotient space

$$\mathrm{Coker}(T) := H_2/\overline{\mathrm{Im}(T)}. \tag{1.4.55}$$

Definition 1.4.14. We say a bounded linear map $T : H_1 \to H_2$ is a **Fredholm operator** if its kernel and cokernel are finite dimensional and its range is closed. The index of the Fredholm operator is defined by

$$\mathrm{ind}(T) := \dim \mathrm{Ker}\, T - \dim \mathrm{Coker}\, T. \tag{1.4.56}$$

Lemma 1.4.15. *Let $P : H_1 \to H_2$ and $Q : H_2 \to H_1$ be bounded linear maps such that $QP = 1 - S_1$ and $PQ = 1 - S_2$, where S_1 and S_2 are compact operators. Then P and Q are Fredholm operators.*

Proof. If $\mathrm{Ker}(P)$ is infinite dimensional, we can choose an orthonormal basis $v_1, v_2, \cdots$ of $\mathrm{Ker}\, P$. Since $S_1|_{\mathrm{Ker}\, P} = \mathrm{Id}$, we can not find a convergent subsequence of $\{S_1(v_i) = v_i\}$. It is a contradiction with the fact that S_1 is a compact operator. Thus $\ker P$ is finite dimensional.

Let V be the orthonormal complement of $\overline{\mathrm{Im} P}$ in H_2. Then $V \simeq \mathrm{Coker}(P)$. Let P^* be the adjoint of P. If $v \in \mathrm{Ker}\, P^*$, then for any $u \in H_1$,

$$0 = \langle P^* v, u\rangle = \langle v, Pu\rangle. \tag{1.4.57}$$

Thus $v \in V$. If $v \in V$, by (1.4.57) $v \in \mathrm{Ker}\, P^*$. So we have

$$\mathrm{Ker}\, P^* \simeq \mathrm{Coker}\, P. \tag{1.4.58}$$

Since S_2 is compact, S_2^* is compact[3]. Since $Q^* P^* = \mathrm{Id} - S_2^*$, we have $\dim(\mathrm{Coker}\, P) = \mathrm{Ker}(P^*) < +\infty$.

At last, we prove that $\mathrm{Im} P$ is closed. Let $v_k = Pu_k$, $k \in \mathbb{N}$ be a sequence such that $v_k \to v$ in H_2. We need to prove that $v = Pu$ for some $u \in H_1$. We may assume that $u_k \in (\mathrm{Ker}\, P)^{\perp}$.

We claim that $\{u_k\}$ is bounded. Otherwise, by passing to a subsequence, we can assume that $\|u_k\| \to \infty$. So $P(u_k/\|u_k\|) = v_k/\|u_k\| \to 0$. Since S_1 is compact, by passing to a subsequence, $\lim_{k\to\infty}(u_k/\|u_k\|) = \lim_{k\to\infty} S_1(u_k/\|u_k\|) = w$. Note that $\|w\| = 1$. However, by continuity, $Pw = 0$. Since $P|_{(\mathrm{Ker}\, P)^{\perp}}$ is injective, $w = 0$. It is a contradiction. Therefore, $\{u_k\}$ is bounded.

Since $\{u_k\}$ is bounded, by passing to a subsequence, $S_1 u_k \to u_\infty$. Since $Qv_k \to Qv$ and $Qv_k = QPu_k = u_k - S_1 u_k$, we have $PQv = \lim_{k\to+\infty}(Pu_k - PS_1 u_k) = v - Pu_\infty$. Thus $v = P(u_\infty + Qv) \in \mathrm{Im} P$. Therefore $\mathrm{Im} P$ is closed and P is Fredholm.

By symmetry, Q is also Fredholm.

The proof of Lemma 1.4.15 is completed. □

Theorem 1.4.16. *Assume that M is compact. Let $P \in \Psi\mathrm{DO}_m(E, F)$ be an elliptic operator. Then for each $s \in \mathbb{R}$, $P_s : \mathbf{H}^s(E) \to \mathbf{H}^{s-m}(F)$ is Fredholm and* $\mathrm{ind}(P_s)$ *is independent of $s \in \mathbb{R}$.*

Proof. By Rellich Theorem, the smoothing operator is compact. Thus by Lemma 1.4.15, P_s is Fredholm.

By Theorem 1.4.6 (3), $\mathrm{Ker}\, P$ consists of smooth sections. Thus its dimension is independent of s. The same is for $\dim \mathrm{Coker}\, P = \dim \mathrm{Ker}\, P^*$.

The proof of Theorem 1.4.16 is completed. □

Since $\mathrm{ind}(P_s)$ is independent of s, we denote it by $\mathrm{ind}(P)$.

Remark that the famous Atiyah-Singer index theorem explains $\mathrm{ind}(P)$ as a topological formula.

[3]"Functional Analysis" by Zhang gongqing, Theorem 4.1.3.